

# Multiple orthogonal polynomials in random matrix theory

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**Abstract.** Multiple orthogonal polynomials are a generalization of orthogonal polynomials in which the orthogonality is distributed among a number of orthogonality weights. They appear in random matrix theory in the form of special determinantal point processes that are called multiple orthogonal polynomial (MOP) ensembles. The correlation kernel in such an ensemble is expressed in terms of the solution of a Riemann-Hilbert problem, that is of size  $(r + 1) \times (r + 1)$  in the case of  $r$  weights.

A number of models give rise to a MOP ensemble, and we discuss recent results on models of non-intersecting Brownian motions, Hermitian random matrices with external source, and the two matrix model. A novel feature in the asymptotic analysis of the latter two models is a vector equilibrium problem for two or more measures, that describes the limiting mean eigenvalue density. The vector equilibrium problems involve both an external field and an upper constraint.

**Mathematics Subject Classification (2000).** Primary 42C05; Secondary 15A52, 31A15, 60C05, 60G55.

**Keywords.** Multiple orthogonal polynomials, non-intersecting Brownian motion, random matrices with external source, two matrix model, vector equilibrium problems, Riemann-Hilbert problem, steepest descent analysis.

## 1. Introduction

**1.1. Random matrix theory.** The Gaussian Unitary Ensemble (GUE) is the most prominent and most studied ensemble in random matrix theory. It is a probability measure on  $n \times n$  Hermitian matrices for which the joint eigenvalue probability density function (p.d.f.) has the explicit form

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^n e^{-\frac{n}{2} x_j^2} \quad (1)$$

where  $Z_n$  is an explicitly known constant. The density (1) can be analyzed with the help of Hermite polynomials. Due to this connection with classical orthogonal

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\*The author was supported in part by FWO-Flanders project G.0427.09, by K.U. Leuven research grant OT/08/33, by the Belgian Interuniversity Attraction Pole P06/02, and by grant MTM2008-06689-C02-01 of the Spanish Ministry of Science and Innovation.

polynomials many explicit calculations can be done, both for finite  $n$  and in the limit  $n \rightarrow \infty$ , see [40]. In particular it leads to a description of the limiting behavior of eigenvalues on the global (macroscopic) scale as well as on the local (microscopic) scale. The global scale is given by the well-known Wigner semi-circle law

$$\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \quad -2 \leq x \leq 2, \quad (2)$$

in the sense that for eigenvalues  $x_1, \dots, x_n$  taken from (1), the empirical eigenvalue distribution  $\frac{1}{n} \sum_{j=1}^n \delta(x_j)$  converges weakly to  $\rho(x)$  almost surely as  $n \rightarrow \infty$ .

The local scale is characterized by the sine kernel

$$\mathcal{S}(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)} \quad (3)$$

in the bulk. This means that for any given  $x^* \in (-2, 2)$  and any fixed  $m \in \mathbb{N}$ , the  $m$ -point correlation function (i.e., the marginal distribution)

$$\begin{aligned} R_{m,n}(x_1, \dots, x_m) \\ = \frac{n!}{(n-m)!} \int_{\mathbb{R}^{n-m}} \left[ \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^n e^{-\frac{1}{2} n x_j^2} \right] dx_{m+1} \cdots dx_n \end{aligned} \quad (4)$$

has the scaling limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{[\rho(x^*)n]^m} R_{m,n} \left( x^* + \frac{x_1}{\rho(x^*)n}, \dots, x^* + \frac{x_m}{\rho(x^*)n} \right) \\ = \det [\mathcal{S}(x_i, x_j)]_{1 \leq i, j \leq m}. \end{aligned} \quad (5)$$

At the edge points  $\pm 2$  the sine kernel (3) is replaced by the Airy kernel

$$\mathcal{A}(x, y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y} \quad (6)$$

and a scaling limit as in (5) (with scaling factor  $cn^{2/3}$  instead of  $\rho(x^*)n$ ) holds for  $x^* = \pm 2$ . This result leads in particular to the statement about the largest eigenvalue

$$\lim_{n \rightarrow \infty} \text{Prob} \left( \max_{1 \leq k \leq n} x_k < 2 + \frac{t}{cn^{2/3}} \right) = \det [I - \mathcal{A}_{(t, \infty)}] \quad (7)$$

where  $\mathcal{A}$  is the Airy kernel (6) and the determinant is the Fredholm determinant of the integral operator with Airy kernel acting on  $L^2(t, \infty)$ . The limiting distribution (7) is the famous Tracy-Widom distribution named after the authors of the seminal work [45] in which the right-hand side of (7) is expressed in terms of the Hastings-McLeod solution of the Painlevé II equation.

These basic results of random matrix theory have been extended and generalized in numerous directions. Within the theory of random matrices, they have been generalized to ensembles with unitary, orthogonal and symplectic symmetry and to non-invariant ensembles (Wigner ensembles). The distribution functions of random matrix theory also appear in many other probabilistic models that have no apparent connection with random matrices (models of non-intersecting paths, tiling models, and stochastic growth models), see e.g. [8], [31].

Mehta's book [40] is the standard reference on random matrix theory. The book of Deift [22] has been very influential in introducing Riemann-Hilbert techniques into the study of random matrices. In recent years, a number of new monographs appeared [2], [6], [16], [23], [29] that cover the various aspects of the theory of random matrices.

**1.2. Unitary ensembles and orthogonal polynomials.** One direction within random matrix theory is the study of ensembles of the form

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM \quad (8)$$

defined on  $n \times n$  Hermitian matrices  $M$ , which reduces to the GUE in case  $V(x) = \frac{1}{2}x^2$ . The ensembles (8) have the property of unitary invariance and are called unitary ensembles. The eigenvalues have the p.d.f.

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^n e^{-nV(x_j)} \quad (9)$$

with a different normalizing constant  $Z_n$ . [Throughout, we use  $Z_n$  to denote a normalizing constant, which may be different from one formula to the next.]

Again explicit calculations can be done due to the connection with orthogonal polynomials [23], [40]. For a given  $n$ , we consider the monic polynomial  $P_{k,n}$  of degree  $k$  that satisfies

$$\int_{-\infty}^{\infty} P_{k,n}(x) x^j e^{-nV(x)} dx = h_{k,n} \delta_{j,k}, \quad j = 0, \dots, k.$$

Then (9) is a determinantal point process [2], [44], with kernel

$$K_n(x, y) = \sqrt{e^{-nV(x)}} \sqrt{e^{-nV(y)}} \sum_{k=0}^{n-1} \frac{P_{k,n}(x) P_{k,n}(y)}{h_{k,n}} \quad (10)$$

which means that for every  $m \in \mathbb{N}$  the  $m$ -point correlation functions, defined as in (4), have the determinantal form

$$\det [K_n(x_i, x_j)]_{i,j=1,\dots,m}.$$

As  $n \rightarrow \infty$ , the limiting mean eigenvalue density

$$\rho(x) = \lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x)$$

is no longer Wigner's semi-circle law (2), but instead it is the density  $\rho$  of the probability measure  $\mu$  that minimizes the weighted logarithmic energy

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x) \quad (11)$$

among all probability measures on  $\mathbb{R}$ .

Local eigenvalue statistics, however, have a universal behavior as  $n \rightarrow \infty$ , that is described by the sine kernel (3) in the bulk. Thus for points  $x^*$  with  $\rho(x^*) > 0$  the limit (5) holds true. At edge points of the limiting spectrum the density  $\rho$  typically vanishes as a square root and then the universal Airy kernel (6) appears. For real analytic potentials  $V$  this was proved in [11], [24] using Riemann-Hilbert methods. This was vastly extended to non-analytic potentials in recent works of Lubinsky [38] and Levin and Lubinsky [37], among many others.

**1.3. This paper.** In this paper we present an overview of the work (mainly of the author and co-workers) on multiple orthogonal polynomials and their relation to random matrix theory. Multiple orthogonal polynomials are a generalization of orthogonal polynomials that have their origins in approximation theory (Hermite-Padé approximation), see e.g. [3, 42].

They enter the theory of random matrices via a generalization of (9) which we call a multiple orthogonal polynomial (MOP) ensemble [34]. We present a number of models that give rise to a MOP ensemble, namely the model of non-intersecting Brownian motions, the random matrix model with external source and the two matrix model.

The MOPs are described by a Riemann-Hilbert problem that may be used for asymptotic analysis as  $n \rightarrow \infty$  by extending the Deift-Zhou method of steepest descent [25]. The extensions are non-trivial and involve either an a priori knowledge of an underlying Riemann surface (the spectral curve) or the formulation of a relevant equilibrium problem from logarithmic potential theory [43], which asks for a generalization of the weighted energy functional (11).

The latter approach has been successfully applied to the random matrix model with external source and to the two matrix model, but only in very special cases, as will be discussed at the end of the paper.

## 2. Multiple orthogonal polynomials

**2.1. MOP ensemble.** We will describe here multiple orthogonal polynomials of type II, which we simply call multiple orthogonal polynomials. There is also a dual notion of type I multiple orthogonal polynomials.

Suppose we have a finite number of weight functions  $w_1, \dots, w_r$  on  $\mathbb{R}$  and a multi-index  $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ . Associated with these data is the monic polynomial  $P_{\vec{n}}$  of degree  $|\vec{n}| = n_1 + \dots + n_r$  so that

$$\int_{-\infty}^{\infty} P_{\vec{n}}(x) x^j w_k(x) dx = 0, \quad \text{for } j = 0, \dots, n_k - 1, \quad k = 1, \dots, r. \quad (12)$$

The linear system of equations (12) may not be always uniquely solvable, but in many important cases it is. If  $P_{\vec{n}}$  uniquely exists then it is called the multiple orthogonal polynomial (MOP) associated with the weights  $w_1, \dots, w_r$  and multi-index  $\vec{n}$ .

Existence and uniqueness does hold in the following situation. Assume that

$$\frac{1}{Z_n} \det [f_j(x_k)]_{j,k=1,\dots,n} \left[ \prod_{1 \leq j < k \leq n} (x_k - x_j) \right] \quad (13)$$

is a p.d.f. on  $\mathbb{R}^n$ , where  $n = |\vec{n}|$  and the linear span of the functions  $f_1, \dots, f_n$  is the same as the linear span of the set of functions

$$\{x^j w_k(x) \mid j = 0, \dots, n_k - 1, k = 1, \dots, r\}.$$

So the assumption is that (13) is non-negative for every choice of  $x_1, \dots, x_n \in \mathbb{R}^n$ , and that the normalization constant  $Z_n$  can be taken so that the integral (13) over  $\mathbb{R}^n$  is equal to one. Then the MOP satisfying (12) exists and is given by

$$P_{\vec{n}}(x) = \mathbb{E} \left[ \prod_{j=1}^n (x - x_j) \right].$$

We call a p.d.f. on  $\mathbb{R}^n$  of the form (13) a MOP ensemble, see [34].

**2.2. Correlation kernel and RH problem.** The MOP ensemble (13) is a determinantal point process [44] (more precisely a biorthogonal ensemble [17]) with a correlation kernel  $K_n$  that is constructed out of multiple orthogonal polynomials of type II and type I. It is conveniently described in terms of the solution of a Riemann-Hilbert (RH) problem. This RH problem for MOPs [47] is a generalization of the RH problem for orthogonal polynomials due to Fokas, Its, and Kitaev [28].

The RH problem asks for an  $(r+1) \times (r+1)$  matrix valued function  $Y$  so that

$$\left\{ \begin{array}{l} \bullet Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{(r+1) \times (r+1)} \text{ is analytic,} \\ \bullet Y \text{ has limiting values on } \mathbb{R}, \text{ denoted by } Y_+ \text{ and } Y_-, \text{ where } Y_{\pm}(x) \text{ is} \\ \quad \text{the limit of } Y(z) \text{ as } z \rightarrow x \in \mathbb{R} \text{ with } \pm \operatorname{Im} z > 0, \text{ satisfying} \\ \quad Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & \cdots & w_r(x) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \text{ for } x \in \mathbb{R}, \\ \bullet Y(z) = (I + O(1/z)) \operatorname{diag}(z^n, z^{-n_1}, \dots, z^{-n_r}) \text{ as } z \rightarrow \infty. \end{array} \right. \quad (14)$$

If the MOP  $P_{\vec{n}}$  with weights  $w_1, \dots, w_r$  and multi-index  $\vec{n} = (n_1, \dots, n_r)$  exists then the RH problem (14) has a unique solution. If the MOPs with multi-indices  $\vec{n} - \vec{e}_j$  also exist, where  $\vec{e}_j$  is the  $j$ th unit vector of length  $r$ , then the first column of  $Y$  consists of

$$Y_{1,1}(z) = P_{\vec{n}}(z), \quad Y_{j+1,1}(z) = c_{j,\vec{n}} P_{\vec{n} - \vec{e}_j}(z), \quad j = 1, \dots, r \quad (15)$$

where  $c_{j,\vec{n}}$  is the constant

$$c_{j,\vec{n}} = -2\pi i \left[ \int_{-\infty}^{\infty} P_{\vec{n}-\vec{e}_j}(x) x^{n_j-1} w_j(x) dx \right]^{-1} \neq 0.$$

The other columns of  $Y$  contain Cauchy transforms

$$Y_{j,k+1}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Y_{j,1}(x) w_k(x)}{x-z} dx, \quad j = 1, \dots, r+1, \quad k = 1, \dots, r.$$

It is a remarkable fact that the correlation kernel of the MOP ensemble (13) is expressed as follows in terms of the solution of the RH problem, see [20],

$$K_n(x, y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & \cdots & w_r(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad x, y \in \mathbb{R}. \quad (16)$$

The inverse matrix  $Y^{-1}$  contains MOPs of type I, and the formula (16) is essentially the Christoffel-Darboux formula for multiple orthogonal polynomials.

Besides giving a concise formula for the correlation kernel, the expression (16) for the kernel gives also a possible way to do asymptotic analysis in view of the Deift-Zhou method of steepest descent for RH problems.

### 3. Non intersecting path ensembles

A rich source of examples of determinantal point processes is provided by non-intersecting path ensembles. In special cases these reduce to MOP ensembles.

**3.1. Non-intersecting Brownian motion.** Consider a one-dimensional strong Markov process with transition probability densities  $p_t(x, y)$  for  $t > 0$ . Suppose  $n$  independent copies are given with respective starting values  $a_1 < a_2 < \cdots < a_n$  at time  $t = 0$  and prescribed ending values  $b_1 < b_2 < \cdots < b_n$  at time  $t = T > 0$  that are conditioned not to intersect in the full time interval  $0 < t < T$ . Then by an application of a theorem of Karlin and McGregor [32], the positions of the paths at an intermediate time  $t \in (0, T)$  have the joint p.d.f.

$$\frac{1}{Z_n} \det [p_t(a_j, x_k)]_{1 \leq j, k \leq n} \cdot \det [p_{T-t}(x_k, b_l)]_{1 \leq k, l \leq n}. \quad (17)$$

In a discrete combinatorial setting the result of Karlin and McGregor is known as the Lindstrom-Gessel-Viennot theorem.

The density function (17) is a biorthogonal ensemble, which in very special cases reduces to the form (13) of a MOP ensemble.

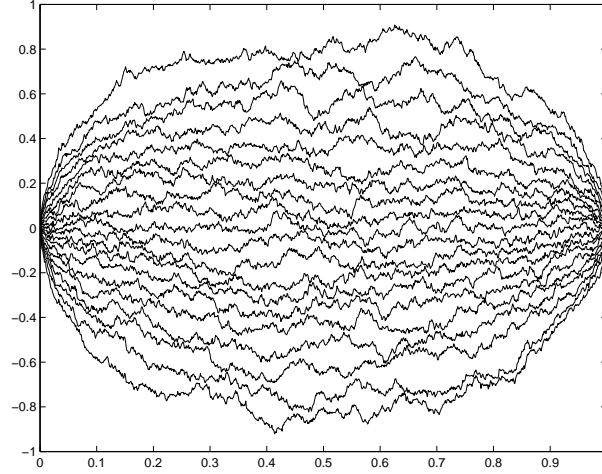


Figure 1. Non-intersecting Brownian bridges starting and ending at 0. At any intermediate time  $t \in (0, 1)$  the positions of the paths have the same distribution as the (appropriately rescaled) eigenvalues of an  $n \times n$  GUE matrix.

An example is the case of Brownian motion (actually Brownian bridges) with the transition probability density

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(x-y)^2}{2t}}, \quad t > 0.$$

In the confluent limit where all  $a_j \rightarrow 0$  and all  $b_l \rightarrow 0$  the p.d.f. (17) turns into

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^n e^{-\frac{T}{2i(T-t)} x_j^2}$$

with a different constant  $Z_n$ . This is up to trivial scaling the same as the p.d.f. (1) for the eigenvalues of an  $n \times n$  GUE matrix.

If however, we let all  $b_l \rightarrow 0$  and choose only  $r$  different starting values, denoted by  $a_1, \dots, a_r$ , and  $n_j$  paths start at  $a_j$ , then (17) turns into a MOP ensemble with weights

$$w_j(x) = e^{-\frac{T}{2i(T-t)} x^2 + \frac{a_j}{t} x}, \quad j = 1, \dots, r, \quad (18)$$

and multi-index  $(n_1, \dots, n_r)$ . This is a multiple Hermite ensemble, since the associated MOPs are multiple Hermite polynomials [5]

**3.2. Non-intersecting squared Bessel paths.** The squared Bessel process is another one-dimensional Markov process which gives rise to a MOP ensemble. The squared Bessel process is a Markov process on  $[0, \infty)$ , depending on a parameter  $\alpha > -1$ , with transition probability density

$$p_t(x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{\alpha/2} e^{-\frac{1}{2t}(x+y)} I_{\alpha} \left(\frac{\sqrt{xy}}{t}\right), \quad x, y > 0,$$

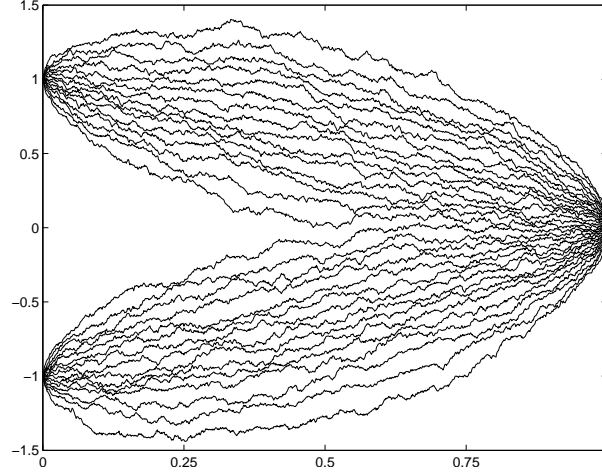


Figure 2. Non-intersecting Brownian bridges starting at two different values and ending at 0. At any time  $t \in (0, 1)$  the positions of the paths have the same distribution as the eigenvalues of an  $n \times n$  GUE matrix with external source. The distribution is a multiple Hermite ensemble with two Gaussian weights (18).

where  $I_\alpha$  is the modified Bessel function of first kind of order  $\alpha$ . In the limit where all  $a_j \rightarrow a > 0$  and  $b_j \rightarrow 0$  the p.d.f. (17) for the positions of the paths at time  $t \in (0, T)$  is a MOP ensemble with two weights

$$w_1(x) = x^{\alpha/2} e^{-\frac{T}{2t(T-t)}x} I_\alpha \left( \frac{\sqrt{ax}}{t} \right)$$

$$w_2(x) = x^{(\alpha+1)/2} e^{-\frac{T}{2t(T-t)}x} I_{\alpha+1} \left( \frac{\sqrt{ax}}{t} \right)$$

and multi-index  $(n_1, n_2)$  where  $n_1 = \lceil n/2 \rceil$  and  $n_2 = \lfloor n/2 \rfloor$ , see [35]. In the limit  $a \rightarrow 0$  this further reduces to an orthogonal polynomial ensemble for a Laguerre weight.

## 4. Random matrix models

The random matrix model with external source, and the two matrix model also give rise to MOP ensembles.

**4.1. Random matrices with external source.** The Hermitian matrix model with external source is the probability measure

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M) - AM)} dM \quad (19)$$

on  $n \times n$  Hermitian matrices, where the external source  $A$  is a given Hermitian  $n \times n$  matrix. This is a modification of the usual Hermitian matrix model, in which the unitary invariance is broken [18], [48].

Due to the Harish-Chandra/Itzykson-Zuber integral formula [30], it is possible to integrate out the eigenvectors explicitly. In case the eigenvalues  $a_1, \dots, a_n$  of  $A$  are all distinct, we obtain the explicit p.d.f.

$$\frac{1}{Z_n} \det [e^{n a_i x_j}]_{1 \leq i, j \leq n} \cdot \prod_{1 \leq j < k \leq n} (x_k - x_j) \cdot \prod_{j=1}^n e^{-n V(x_j)}$$

for the eigenvalues of  $M$ . In case that  $a_1, \dots, a_r$  are the distinct eigenvalues of  $A$ , with respective multiplicities  $n_1, \dots, n_r$ , then the eigenvalues of  $M$  are distributed as a MOP ensemble (13) with weights

$$w_j(x) = e^{-n(V(x) - a_j x)}, \quad j = 1, \dots, r \quad (20)$$

and multi-index  $(n_1, \dots, n_r)$ , see [13]

For the case  $V(x) = \frac{1}{2}x^2$  the external source model (19) is equivalent to the model of non-intersecting Brownian motions with several starting points and one ending point, cf. (18).

#### 4.2. Two matrix model. The Hermitian two matrix model

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M_1) + W(M_2) - \tau M_1 M_2)} dM_1 dM_2 \quad (21)$$

is a probability measure defined on couples  $(M_1, M_2)$  of  $n \times n$  Hermitian matrices. Here  $V$  and  $W$  are two potentials (typically polynomials) and  $\tau \neq 0$  is a coupling constant. The model is of great interest in 2d quantum gravity [21], [30], [33], as it allows for a large class of critical phenomena.

The eigenvalues of the matrices  $M_1$  and  $M_2$  are fully described by biorthogonal polynomials. These are two sequences  $(P_{k,n})_k$  and  $(Q_{j,n})_j$  of monic polynomials,  $\deg P_{k,n} = k$ ,  $\deg Q_{j,n} = j$ , such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_{k,n}(x) Q_{j,n}(y) e^{-n(V(x) + W(y) - \tau xy)} dx dy = 0, \quad \text{if } j \neq k, \quad (22)$$

see e.g. [9], [27], [40], [41].

If  $W$  is a polynomial then the biorthogonality conditions (22) can be seen as multiple orthogonal polynomial conditions with respect to  $r = \deg W - 1$  weights

$$w_{j,n}(x) = e^{-nV(x)} \int_{-\infty}^{\infty} y^j e^{-n(W(y) - \tau xy)} dy, \quad j = 0, \dots, r-1, \quad (23)$$

see [36]. Furthermore, the eigenvalues of  $M_1$  are a MOP ensemble (13) with the weights (23) and multi-index  $\vec{n} = (n_0, \dots, n_{r-1})$  with  $n_j = \lceil n/r \rceil$  for  $j = 0, \dots, q-1$  and  $n_j = \lfloor n/r \rfloor$  for  $j = q, \dots, r-1$  if  $n = pr + q$  with  $p$  and  $0 \leq q < r$  non-negative integers, see [26] for the case where  $W(y) = \frac{y^4}{4}$ .

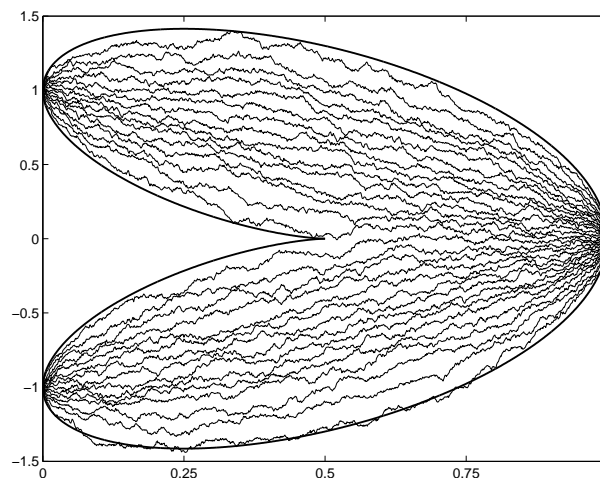


Figure 3. Non-intersecting Brownian bridges starting at two different values and ending at 0. As  $n \rightarrow \infty$ , the paths fill out a heart-shaped domain. Critical behavior at the cusp point is described by the Pearcey kernel (24).

## 5. Large $n$ behavior and critical phenomena

We discuss the large  $n$  behavior in the above described models.

**5.1. Non-intersecting Brownian motion.** In order to have interesting limit behavior as  $n \rightarrow \infty$  in the non-intersecting Brownian motion model we scale the time variables  $T \mapsto 1/n$ ,  $t \mapsto t/n$ , so that  $0 < t < 1$ . In the case of one starting value and one ending value, see Figure 1, the paths will fill out an ellipse as  $n \rightarrow \infty$ .

In the situation of Figure 2 the paths fill out a heart-shaped region as  $n \rightarrow \infty$ , as shown in Figure 3. New critical behavior appears at the cusp point where the two groups of paths come together and merge into one.

Around the critical time the correlation kernels have a double scaling limit, which is given by the one-parameter family of Pearcey kernels

$$\mathcal{P}(x, y; b) = \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - bp(x)q(y)}{x - y}, \quad b \in \mathbb{R}, \quad (24)$$

where  $p$  and  $q$  are solutions of the Pearcey equations  $p'''(x) = xp(x) - bp'(x)$  and  $q'''(y) = yq(y) + bq'(y)$ . This kernel was first identified by Brézin and Hikami [18] who also gave the double integral representation

$$\mathcal{P}(x, y) = \frac{1}{(2\pi i)^2} \int_C \int_{-\infty}^{i\infty} e^{-\frac{1}{4}s^4 + \frac{b}{2}s^2 - ys + \frac{1}{4}t^4 - \frac{b}{2}t^2 + xt} \frac{ds dt}{s - t} \quad (25)$$

where the contour  $C$  consists of the rays from  $\pm\infty e^{i\pi/4}$  to 0 and the rays from 0 to  $\pm\infty e^{-i\pi/4}$ .

Consideration of multiple times near the critical time leads to an extended Pearcey kernel and the Pearcey process given by Tracy and Widom [46].

As already noted above, the model of non-intersecting Brownian motion with two starting points and one ending point is related to the Gaussian random matrix model with external source

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(\frac{1}{2} M^2 - AM)} dM, \quad (26)$$

with external source

$$A = \operatorname{diag}(\underbrace{a, \dots, a}_{n/2 \text{ times}}, \underbrace{-a, \dots, -a}_{n/2 \text{ times}}). \quad (27)$$

In this setting the critical  $a$ -value is  $a_{crit} = 1$  and the Pearcey kernel (24) arises as  $n \rightarrow \infty$  with  $a = 1 + \frac{b}{2\sqrt{n}}$ . In [15] this was studied with the use of the Riemann-Hilbert problem (14) for multiple Hermite polynomials with two weights  $e^{-n(\frac{1}{2}x^2 \pm ax)}$ . The asymptotic analysis as  $n \rightarrow \infty$  was done with an extension of the Deift-Zhou method of steepest descent [25] to the case of a  $3 \times 3$  matrix valued RH problem. See also [14] and [4] for a steepest descent analysis of the RH problem in the non-critical regimes  $a > 1$  and  $0 < a < 1$ , respectively.

Another interesting asymptotic regime is the model of non-intersecting Brownian motion with outliers. In this model a rational modification of the Airy kernel appears that was first described in [7] in the context of complex sample covariance matrices, see also [1].

**5.2. Random matrices with external source.** If  $V$  is quadratic in the random matrix model with external source (19) then this model can be mapped to the model of non-intersecting Brownian motions. Progress on this model beyond the quadratic case is due to McLaughlin [39] who found the spectral curve for the quartic potential  $V(x) = \frac{1}{4}x^4$  and for  $a$  sufficiently large (again  $A$  is as in (27)).

A method based on a vector equilibrium problem was introduced recently by Bleher, Delvaux and Kuijlaars [10]. The vector equilibrium problem extends the equilibrium problem for the weighted energy (11) that is important for the unitary ensembles and which is crucial in the steepest descent analysis of the RH problem for orthogonal polynomials [24].

In [10] it is assumed that  $V$  is an even polynomial, and that  $A$  is again given as in (27). The vector equilibrium problem involves two measures  $\mu_1$  and  $\mu_2$ , and it asks to minimize the energy functional

$$\begin{aligned} & \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) + \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y) \\ & - \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y) + \int (V(x) - a|x|) d\mu_1(x) \end{aligned} \quad (28)$$

where  $\mu_1$  is on  $\mathbb{R}$  with  $\int d\mu_1 = 1$ ,  $\mu_2$  is on  $i\mathbb{R}$  (the imaginary axis) with  $\int d\mu_2 = 1/2$ , and in addition  $\mu_2 \leq \sigma$ , where  $\sigma$  is the measure on  $i\mathbb{R}$  with constant density

$$\frac{d\sigma}{|dz|} = \frac{a}{\pi}. \quad (29)$$

There is a unique minimizer, and the density  $\rho_1$  of the measure  $\mu_1$  is the limiting mean eigenvalue density

$$\rho_1(x) = \lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x)$$

where  $K_n$  is the correlation kernel of the MOP ensemble with weights  $e^{-n(V(x) \pm ax)}$ . The RH problem (14) is analyzed in the large  $n$  limit with the Deift/Zhou steepest descent method in which the minimizers from the vector equilibrium problem play a crucial role.

The upper constraint  $\mu_2 \leq \sigma$  is not active for large enough  $a$  and in that case the support of  $\mu_1$  has a gap around 0. For smaller values of  $a$  the constraint  $\sigma$  is active along an interval  $[-ic, ic]$ ,  $c > 0$ , on the imaginary axis. Critical phenomena take place when either the constraint becomes active, or the gap around 0 closes, or both. If one of these two phenomena happens, then this generically will be a phase transition of the Painlevé II type that was described in the unitary matrix model in [12] and [19]. If the two phenomena happen simultaneously then this is expected to be phase transition of the Pearcey type which, if true, would be a confirmation of the universality of the Pearcey kernels (24) at the closing of a gap [18].

Both kinds of transitions are valid in the external source model with even quartic potential  $V(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$ , see [10]. For the particular value  $t = \sqrt{3}$ , there is a passage from the Painlevé II transition (for  $t > \sqrt{3}$ ) to the Pearcey transition (for  $t < \sqrt{3}$ ). The description of the phase transition for  $t = \sqrt{3}$  remains open.

**5.3. Two matrix model.** In [26] Duits and Kuijlaars applied the steepest descent analysis to the RH problem for the two matrix model (21) with quartic potential

$$W(y) = \frac{1}{4}y^4 \quad (30)$$

and for  $V$  an even polynomial. The corresponding MOP ensemble has three weights of the form (23) and the RH problem (14) is of size  $4 \times 4$ . Again a vector equilibrium problem plays a crucial role.

The vector equilibrium problem in [26] involves three measures  $\mu_1$ ,  $\mu_2$  and  $\mu_3$ . It asks to minimize the energy functional

$$\begin{aligned} & \sum_{j=1}^3 \iint \log \frac{1}{|x-y|} d\mu_j(x) d\mu_j(y) \\ & - \sum_{j=1}^2 \iint \log \frac{1}{|x-y|} d\mu_j(x) d\mu_{j+1}(y) + \int (V(x) - \frac{3}{4}|\tau x|^{4/3}) d\mu_1(x) \end{aligned} \quad (31)$$

among measures  $\mu_1$  on  $\mathbb{R}$  with  $\int d\mu_1 = 1$ ,  $\mu_2$  on  $i\mathbb{R}$  with  $\int d\mu_2 = 2/3$  and  $\mu_3$  on  $\mathbb{R}$  with  $\int d\mu_3 = 1/3$ . In addition  $\mu_2 \leq \sigma$  where  $\sigma$  is a given measure on  $i\mathbb{R}$  with density

$$\frac{d\sigma}{|dz|} = \frac{\sqrt{3}}{2\pi} |\tau|^{4/3} |z|^{1/3}, \quad z \in i\mathbb{R}. \quad (32)$$

There is a unique minimizer and the density  $\rho_1$  of the first measure  $\mu_1$  is equal to the limiting mean eigenvalue density of the matrix  $M_1$  in the two matrix model. In addition, the usual scaling limits (sine kernel and Airy kernel) are valid in the local eigenvalue regime, see [26]. However there is no new critical behavior in the two matrix model with  $W$  is given by (30).

New multicritical behavior is predicted in [21] for more general potentials. For the more general quartic potential  $W(y) = \frac{1}{4}y^4 - \frac{t}{2}y^2$  an approach based on a modification of the vector equilibrium problem (31) is under current investigation.

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